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THE L^p DIRICHLET PROBLEM FOR SECOND-ORDER, NON-DIVERGENCE FORM OPERATORS: SOLVABILITY AND PERTURBATION RESULTS

MARTIN DINDOŠ AND TREVEN WALL

ABSTRACT. We establish Dahlberg's perturbation theorem for non-divergence form operators $\mathcal{L} = A\nabla^2$. If \mathcal{L}_0 and \mathcal{L}_1 are two operators on a Lipschitz domain such that the L^p Dirichlet problem for the operator \mathcal{L}_0 is solvable for some $p \in (1, \infty)$ and the coefficients of the two operators are sufficiently close in the sense of Carleson measure, then the L^p Dirichlet problem for the operator \mathcal{L}_1 is solvable for the same p . This is an improvement of the A_∞ version of this result proved by Rios in [10]. As a consequence we also improve a result from [4] for the L^p solvability of non-divergence form operators (Theorem 3.2) by substantially weakening the condition required on the coefficients of the operator. The improved condition is exactly the same one as is required for divergence form operators $\mathcal{L} = \operatorname{div} A\nabla$.

1. INTRODUCTION

This paper is a continuation of a long line of work, most recently advanced in [4], on the solvability of the L^p Dirichlet problem for elliptic operators with rough coefficients on Lipschitz domains with small constants. Here, we extend the results to non-divergence form operators satisfying a certain oscillatory Carleson condition which was left open in [4].

In particular, the non-divergence form results in [4] require a condition on the gradient of the coefficient matrix, since their results arise from considering the divergence form case first. They then change a non-divergence form operator to divergence form by allowing first order terms.

Throughout this paper, the operators \mathcal{L} which we consider are second-order, linear, uniformly elliptic and in non-divergence form. Precisely, $\mathcal{L} = a^{ij}(x)\partial_{ij}$ (we use here and throughout the paper the usual summation convention), where $A(x) = (a^{ij}(x))_{i,j}$ is a symmetric matrix with ellipticity constant $0 < \lambda < \infty$ such that for all $x, \xi \in \mathbb{R}^n$,

$$\lambda |\xi|^2 \leq \xi^t A(x) \xi \leq \lambda^{-1} |\xi|^2. \quad (1.1)$$

We assume throughout that $n \geq 2$.

The problem under consideration is the Dirichlet problem

$$\begin{aligned} \mathcal{L}u &= 0 && \text{in } D \\ u &= g && \text{on } \partial D, \end{aligned} \quad (1.2)$$

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where $D \subset \mathbb{R}^n$ is a bounded Lipschitz domain.

It is a fairly difficult task to even define the notion of a solution to the equation (1.2). Recall that in the divergence form case one can use Peron's method to construct a solution for coefficients that are merely bounded and measurable. This is not the case in our situation. For this reason we postulate what we mean by solving the *continuous Dirichlet problem*, denoted \mathcal{CD} , following [10] and [11]:

Definition 1.1. (Continuous Dirichlet problem, \mathcal{CD}) Given an operator \mathcal{L} we say that the continuous Dirichlet problem is uniquely solvable in D (and we say \mathcal{CD} holds for \mathcal{L}) if for every continuous function g on ∂D there exists a unique solution u of (1.2) such that $u \in C(\overline{D}) \cap W_{loc}^{2,p}(D)$ for some $1 \leq p \leq \infty$.

From the results in [1], if the coefficients a^{ij} are in VMO (see section 2 for the definition) and $g \in C(\partial D)$, then (1.2) has a unique solution $u_g \in C(\overline{D}) \cap W_{loc}^{2,p}(D)$ for all p , $1 < p < \infty$. By approximation, one can extend this result to allow for coefficients in BMO_{ϱ_0} (also defined in section 2) in a restricted range of p : $1 < p < p_0(\varrho_0)$ (c.f. [12]).

Definition 1.2. (L^p Dirichlet problem, \mathcal{D}_p). We say that the Dirichlet problem is solvable for \mathcal{L} in L^p on D (or that \mathcal{D}_p holds for \mathcal{L} on D), for $1 < p < \infty$, if \mathcal{CD} holds for \mathcal{L} and there is a constant C (depending only on \mathcal{L} , λ , n , D and p) such that for all $g \in C(\partial D)$, the \mathcal{CD} solution to (1.2) (which we shall denote by u_g) satisfies

$$\|Nu_g\|_{L^p(\partial D)} \leq C\|g\|_{L^p(\partial D)}, \quad (1.3)$$

where N is the non-tangential maximal operator (see below). Unless explicitly stated, the assumed measure on ∂D is σ , standard surface measure.

Here the non-tangential maximal operator N (when necessary, N_α) is defined as

$$N_\alpha u(Q) = \sup_{x \in \Gamma_\alpha(Q)} |u(x)|,$$

where $\Gamma_\alpha(Q)$ denotes a truncated cone interior to D of aperture α based at $Q \in \partial D$, i.e.,

$$\Gamma_\alpha(Q) = \{x \in D : |x - Q| \leq (1 + \alpha)\delta(x)\} \cap B_{r^*}(Q).$$

Throughout, $\alpha > \alpha^*(D) > 0$, with α^* and $r^* > 0$ determined by the Lipschitz character of the domain and its size. Finally, $\delta(x) := \text{dist}(x, \partial D)$; $B_r(x)$ denotes the ball of radius r centered at x .

In this paper we consider two fundamental questions. The first one is whether the L^p solvability can be perturbed, that is, if \mathcal{L}_0 and \mathcal{L}_1 are two operators close in some sense, under what conditions does the solvability of the L^p Dirichlet problem for one operator imply the same for the other? This question has a long history in the case of second order elliptic divergence form operators. For our purposes the papers [2] and [6] are of particular importance. Operators with first order terms are considered in [7].

The non-divergence form case has a considerably shorter history since new difficult issues arise. In particular, the non-uniqueness of so-called *weak solutions* causes trouble in the most general case (see [9] and [13]).

However, assuming, as we do, that the coefficients of the non-divergence form operators considered have a small BMO norm, one can establish the existence of *strong solutions* (i.e. solutions in $W_{loc}^{2,p}(D)$, c.f [1]). These are the solutions we consider in this paper.

The papers [10] and [11] have made very good progress in settling the question whether results that hold in the divergence form case extend to non-divergence form operators. In particular, these papers show that if the elliptic measure of an operator \mathcal{L}_0 is in the Muckenaupt $A_\infty(d\sigma)$ class, then so is the elliptic measure of an operator \mathcal{L}_1 under the same assumptions as in [6]. This implies that L^p solvability of the operator \mathcal{L}_0 gives L^q solvability of the operator \mathcal{L}_1 (for q potentially much larger than p). The paper [11] also considers first order terms (drift terms).

In our Theorem 3.1 we settle the question of whether q can be taken to be the same as p , and the answer is affirmative if the coefficients of the considered operators are sufficiently close in the sense of Carleson measure. Analogous results for divergence form operators have been established before ([2] and [6]). We do not consider first order terms as [11] does; however, this can be done, as all the necessary ingredients are already in place. We choose not to do it here to make our already very technical exposition more readable.

The second fundamental question we settle here is the question of finding a broad condition on coefficients of the non-divergence form operator that guarantee L^p solvability. Again, the case of divergence form operators serves as a model. There are two particularly important results to mention here. In [8] it was established that under the assumption that $t|\nabla A|^2$ is a Carleson measure, the L^p Dirichlet problem is solvable. In [4] this condition was relaxed (the gradient is replaced by an oscillation-type condition), and it was also shown there that given $p \in (1, \infty)$ the L^p solvability depends on the norm of the Carleson measure. If the norm is small, the L^p solvability for particular p holds. Moreover, since first order terms are also considered, the results in [8] and [4] do apply to non-divergence form operators, but only under the stronger gradient condition $t|\nabla A|^2$.

The missing piece to prove this under the weaker “oscillation condition” is a strong perturbation theorem for non-divergence form operators which we establish here. Hence the result in [4] is substantially improved. This is formulated in Theorem 3.2. As previously mentioned, the weaker A_∞ version of this result is already done in [11].

The structure of this paper is as follows: section 2 contains a few definitions needed to formulate two main results, which is done in section 3. Section 4 expands the list of definitions and introduces a few technical preliminaries. Section 5 contains the proof of the perturbation result and section 6 is dedicated to the L^p solvability under the Carleson condition on the coefficients of our operator. Finally, sections 7 and 8 contain the proofs of two auxiliary lemmas.

2. BASIC DEFINITIONS

Given $f \in L^1_{loc}(\mathbb{R}^n)$, let

$$\eta(r, x) = \eta_f(r, x) = \sup_{s \leq r} \frac{1}{|B_s(x)|} \int_{B_s(x)} |f(y) - f_{B_s(x)}| dy,$$

where f_E is the average value of f on E . Then $f \in \text{BMO}(\mathbb{R}^n)$ (i.e., f has *bounded mean oscillation*) if $\eta \in L^\infty(\mathbb{R}^+, \mathbb{R}^n)$. Moreover, $\|f\|_{\text{BMO}} = \|\eta_f\|_{L^\infty(\mathbb{R}^+, \mathbb{R}^n)}$.

Let $\eta(r) = \|\eta(r, \cdot)\|_{L^\infty(\mathbb{R}^n)}$. We say $f \in \text{VMO}(\mathbb{R}^n)$ (f has *vanishing mean oscillation*) if $\lim_{r \rightarrow 0^+} \eta(r) = 0$. Finally, a function $f \in \text{BMO}_\varrho(\mathbb{R}^n)$ if $\eta(r) \in \Phi(\varrho)$, where $\Phi(\varrho)$ is the collection of all non-decreasing functions $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

there exists a $\zeta > 0$ such that $\eta(r) \leq \varrho$ for all $r < \zeta$. These spaces (BMO, VMO, BMO_ϱ) can be restricted to a Borel set G using standard methods.

The setting for our work is a Lipschitz domain D . A bounded, connected domain $D \subset \mathbb{R}^n$ is called a *Lipschitz domain* if there is a finite collection $\{(I_i, \phi_i)\}$ of right circular cylinders I_i and Lipschitz functions ϕ_i ($\phi_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, and there is an $L > 0$ such that for all $x, y \in \mathbb{R}^{n-1}$, $|\phi(x) - \phi(y)| \leq L|x - y|$) such that the following hold:

- (i) The collection of cylinders $\{I_i\}$ covers the boundary, ∂D , of D .
- (ii) The bases of the cylinders have positive distance from ∂D .
- (iii) Corresponding to each pair (I_i, ϕ_i) , there is a coordinate system (x, s) with $x \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ such that the x -axis is parallel to the axis of I_i and such that $I_i \cap D = \{(x, s) : s > \phi_i(x)\} \cap I_i$ and $I_i \cap \partial D = \{(x, s) : s = \phi_i(x)\} \cap I_i$.

Without loss of generality, we will assume that D is contained within the unit ball centered at the origin of \mathbb{R}^n and that D contains the origin, i.e., we assume $D \subset B_1(0)$ with $0 \in D$.

For points $Q \in \partial D$ and $r > 0$, we denote the boundary ball of radius r at Q by $\Delta_r(Q) = B_r(Q) \cap \partial D$. The *Carleson region* $T_r(Q)$ above $\Delta_r(Q)$ is given by $T_r(Q) = B_r(Q) \cap D$. We say that a measure μ on D is a *Carleson measure* if there is an $M < \infty$ such that

$$\sup_{r>0, Q \in \partial D} \frac{\mu(T_r(Q))}{\sigma(\Delta_r(Q))} = M.$$

3. MAIN RESULTS

The following perturbation theorem is modelled on Dahlberg's theorem (Theorem 1 in [2], re-proven as Theorem 2.18 in [6]).

Theorem 3.1. *Consider operators $\mathcal{L}_0, \mathcal{L}_1$, with $\mathcal{L}_k = a_k^{ij}(x)\partial_{ij}$ on a Lipschitz domain D , $\varepsilon(x) = \left(a_0^{ij}(x) - a_1^{ij}(x)\right)_{i,j}$ and $\mathbf{a}(x) = \sup_{z \in B_{\frac{\delta(x)}{2}}(x)} |\varepsilon(z)|$. Let $\lambda > 0$ be the ellipticity constant of the operator \mathcal{L}_0 , and let*

$$\sup_{Q \in \partial D, r>0} \frac{1}{\sigma(\Delta_r(Q))} \int_{T_r(Q)} \frac{\mathbf{a}^2(x)}{\delta(x)} dx = \varepsilon_0 < \infty. \quad (3.4)$$

Assume that the L^p Dirichlet problem is solvable for the operator \mathcal{L}_0 with a constant $C_p > 0$ in the estimate (1.3) for some $1 < p < \infty$.

There exist constants $\varrho_0 > 0$ (independent of p) and $M = M(p, D, \lambda, C_p, \varrho) > 0$ such that if $a_0^{ij} \in \text{BMO}_\varrho$ with $\varrho < \varrho_0$, and if $\varepsilon_0 < M$, then the L^p Dirichlet problem is solvable for the operator \mathcal{L}_1 .

Remark 1. This theorem is a direct improvement of Theorem 1.1 in [10] and Theorem 2.1 in [11] where a statement of the type \mathcal{D}_p for $\mathcal{L}_0 \implies \mathcal{D}_q$ for \mathcal{L}_1 was established with $q \gg p$.

Remark 2. It suffices to assume that the condition $\varepsilon_0 < M$ in Theorem 3.1 only holds for all Carleson regions T_r such that $r \leq r_0$ for some $r_0 > 0$. This is due to the comparability of the elliptic measures of two operators whose coefficients are the same near the boundary (see Lemma 2.15 of [10]).

Remark 3. The theorem can be formulated on more general domains. In fact, we never explicitly use that the boundary of D has a graph-like structure. The minimal

geometric structure needed is that D be a chord-arc domain and non-tangetially accessible.

Remark 4. The number ϱ_0 is chosen such that if $a^{ij} \in \text{BMO}_\varrho$ for any $\varrho < \varrho_0$, \mathcal{CD} holds for the operator $\mathcal{L} = a^{ij}\partial_{ij}$ on D and so that the value of $p_0(\varrho_0)$ referenced in the discussion after Definition 1.1 satisfies $p_0(\varrho_0) > 2$.

Theorem 3.2. *Let $1 < p < \infty$, let $0 < \lambda < \infty$ be a fixed ellipticity constant, and let D be a Lipschitz domain with Lipschitz constant L . Let $\mathcal{L} = a^{ij}\partial_{ij}$ be an elliptic operator with ellipticity constant λ .*

If

$$\sup \left\{ \frac{|a^{ij}(x) - \text{avg}(a^{ij}(z))|^2}{\delta(x)} : x \in B_{\frac{\delta(z)}{2}}(z) \right\}$$

is the density of a Carleson measure in D with Carleson constant M , then there is a constant $C(p, \lambda) > 0$ such that if $L < C(p, \lambda)$ and $M < C(p, \lambda)$, the Dirichlet problem \mathcal{D}_p is solvable for the operator \mathcal{L} . [Here $\text{avg}(a^{ij}(z))$ is the average of the coefficient a^{ij} over the ball $B_{\delta(z)/2}(x)$.]

This is a substantial improvement of Corollary 2.5 in [4], in the spirit of Corollary 2.3 in the same paper for divergence form operators. This also improves Theorem 2.4 in [11]. Note that we do not consider drift terms as [11] does.

4. NOTATION AND TECHNICAL PRELIMINARIES

To enhance the readability of this paper, we have kept our notation the same as in [10]. We rely heavily on certain results from [10] in the technical part of this paper.

Throughout the paper, we use $A \lesssim B$ to mean there is a constant C , depending on, at most, n, λ, η and D such that $A \leq CB$; similarly for $A \gtrsim B$. If $A \lesssim B$ and $A \gtrsim B$, then we say $A \approx B$.

Let \mathcal{L} be an elliptic operator for which \mathcal{CD} holds. By the maximum principle, the mapping $g \mapsto u_g(x)$ is a positive linear functional on $C(\partial D)$ for each fixed $x \in D$. The Riesz representation theorem then gives a unique regular positive Borel measure ω^x on ∂D such that

$$u(x) = \int_{\partial D} g(Q) d\omega^x(Q).$$

This measure is called the harmonic measure for \mathcal{L} on D .

Given a non-decreasing function η , we denote by $O(\lambda, \eta)$ the class of operators $\mathcal{L} = a^{ij}\partial_{ij}$ with symmetric coefficients satisfying (1.1) such that $a^{ij} \in \text{BMO}(\mathbb{R}^n)$ with BMO modulus of continuity η in D . We use $O(\lambda)$ if there is no restriction on the regularity of \mathcal{L} .

The theory of weights plays an important role in what follows. Given a p , $1 < p < \infty$, and two measures μ and ν on ∂D , if μ is absolutely continuous with respect to ν , let $k = \frac{\partial \mu}{\partial \nu}$. Then, we say that $\mu \in A_p(d\nu)$ if there is a constant $C < \infty$ such that for all boundary balls Δ (i.e., for some $r > 0$, $Q \in \partial D$, $\Delta = \Delta_r(Q)$),

$$\left(\frac{1}{\nu(\Delta)} \int_{\Delta} k d\nu \right) \left(\frac{1}{\nu(\Delta)} \int_{\Delta} k^{-\frac{1}{p-1}} d\nu \right)^{\frac{1}{p-1}} \leq C. \quad (4.5)$$

We say that $\mu \in RH_p(d\nu)$, the reverse-Hölder class, if there is a constant C such that for all boundary balls Δ ,

$$\left(\frac{1}{\nu(\Delta)} \int_{\Delta} k^p d\nu \right)^{\frac{1}{p}} \leq C \frac{1}{\nu(\Delta)} \int_{\Delta} k d\nu. \quad (4.6)$$

Note that $\mu \in A_p(d\nu)$ if and only if $\nu \in RH_{p'}(d\mu)$, with $p' = \frac{p}{p-1}$. The best constant C in (4.5) is called the A_p “norm” of μ and is denoted $A_p(\mu|d\nu)$. Recall that the assumed measure on ∂D is σ , standard surface measure, so by $\mu \in A_p$, we mean $\mu \in A_p(d\sigma)$. Also, these classes of measures (or weights) are related:

$$\bigcup_{p' > 1} RH_{p'}(d\nu) = \bigcup_{p > 1} A_p(d\nu) =: A_{\infty}(d\nu).$$

A crucial ingredient in what follows is the fundamental theorem relating weights to solutions of elliptic partial differential equations (first proved by Dahlberg in [2]):

Theorem 4.1. *Let ω be the harmonic measure with respect to \mathcal{L} on D , and let μ be a Borel measure on ∂D . Then the following are equivalent:*

- (i) $\omega \in A_{\infty}(d\mu)$.
- (ii) There is a $1 < p < \infty$ such that \mathcal{D}_p holds, i.e.,

$$\|Nu_g\|_{L^p(\partial D, d\mu)} \leq C_p \|g\|_{L^p(\partial D, d\mu)}.$$

- (iii) ω is absolutely continuous with respect to μ and $\omega \in RH_{p'}(d\mu)$ (where, again, $p' = \frac{p}{p-1}$).

Lemma 4.2. (Theorem 2.5 in [10]) *If we let $p \in [n, \infty)$, $w \in A_p$, then there is a constant ϱ_p such that if $\eta \in \Phi(\varrho_p)$ and $\mathcal{L} \in O(\lambda, \eta)$, for any $f \in L^p(D, w)$, there exists a unique $u \in C(\overline{D}) \cap W^{2,p}(D, w)$ such that $\mathcal{L}u = f$ in D and $u = 0$ on ∂D .*

Then, with f and u as in Lemma 4.2, for each $x \in D$, the maximum principle implies that the positive linear functional $f \rightarrow -u(x)$ is bounded on $L^p(D)$. The Riesz representation theorem then gives us the unique non-negative function $G(x, \cdot) \in L^{p'}(D)$ ($p' = \frac{p}{p-1}$) such that

$$u(x) = - \int_D G(x, y) f(y) dy.$$

This is the Green’s function for \mathcal{L} in D .

Definition 4.1. Given $\mathcal{L} \in O(\lambda)$, $v \in L^1_{loc}(D)$ is an *adjoint solution* of \mathcal{L} in D if

$$\int_D v \mathcal{L}\phi dx = 0$$

for all $\phi \in C_c^\infty(D)$. In this case, we write $\mathcal{L}^*v = 0$.

Recall that we are assuming $D \subset B_1(0)$. Pick a point $\bar{x} \in \partial B_9(0)$, and let

$$\wp(y) = G_{\mathcal{L}, B_{10}(0)}(\bar{x}, y) \text{ in } B_{10}(0), \quad (4.7)$$

where $G_{\mathcal{L}, B_{10}(0)}$ is the Green’s function for \mathcal{L} in $B_{10}(0)$. The following technical estimate is quite useful.

Lemma 4.3. (Lemma 2 in [5]). Let $G(x, y)$ be the Green's function in D for $\mathcal{L} \in O(\lambda)$. Then there is a constant r_0 depending on the Lipschitz character of D , such that for all $Q \in \partial D$, $r \leq r_0$, $y \in \partial B_r(Q) \cap \Gamma_1(Q)$ and $x \notin T_{4r}(Q)$, the following holds:

$$\frac{G(x, y)}{\delta(y)^2} \frac{\wp(B(y))}{\wp(y)} \approx \omega^x(\Delta_r(Q)),$$

with \wp as defined in (4.7), $B(y) = B_{\delta(y)/2}(y)$ and $\wp(B(y)) = \int_{B(y)} \wp(y) dy$.

Following Rios, [10], we define a modified area function and non-tangential maximal function which are adapted for the non-divergence form situation.

Definition 4.2. [Area functions] For a function u defined on D , the area function of aperture α , $S_\alpha u$, and the second area function of aperture α , $A_\alpha u$, are defined as

$$S_\alpha u(Q)^2 = \int_{\Gamma_\alpha(Q)} \frac{\delta^2(x)}{\wp(B(x))} |\nabla u(x)|^2 \wp(x) dx, \text{ and}$$

$$A_\alpha u(Q)^2 = \int_{\Gamma_\alpha(Q)} \frac{\delta^4(x)}{\wp(B(x))} |\nabla^2 u(x)|^2 \wp(x) dx,$$

with $\wp(x)$ as in (4.7), $B(x) = B_{\delta(x)/2}(x)$, and $Q \in \partial D$.

We also recall Rios' modified non-tangential maximal function,

$$\left(\tilde{N}_\alpha(v) \right)^2(Q) := \sup_{x \in \Gamma_\alpha(Q)} \int_{B_0(x)} v(y)^2 \frac{\wp(y)}{\wp(B(y))} dy. \quad (4.8)$$

Here, $B_0(x)$ denotes $B(x, \frac{\delta(x)}{6})$.

5. PROOF OF THE PERTURBATION THEOREM 3.1

The structure of our proof owes much to the proof of Theorem 2.18 in [6]. We use S_α , A_α and \tilde{N}_α as defined in the section above.

Let \mathcal{L}_0 and \mathcal{L}_1 be two operators as in Theorem 3.1, and consider any continuous boundary data g . We first establish that \mathcal{CD} holds for \mathcal{L}_1 . We observe that $\|A_0 - A_1\|_{L^\infty(D)}^2 \lesssim \varepsilon_0$. Since $a_0^{ij} \in \text{BMO}_\varrho$ and (3.4) holds with $\varepsilon_0 < M$, $a_1^{ij} \in \text{BMO}_{\varrho+\varepsilon}$, where ε can be arbitrary small (it depends on M). So if M is made small enough, we can ensure that $\varrho + \varepsilon < \varrho_0$. Recall that ϱ_0 is chosen to guarantee that \mathcal{CD} holds for any operator $\mathcal{L}_k = a_k^{ij} \partial_{ij}$, as long as $a_k^{ij} \in \text{BMO}_\varrho$ and $\varrho < \varrho_0$.

Notice also that if λ is the ellipticity constant of \mathcal{L}_0 , one can guarantee that the ellipticity constant of \mathcal{L}_1 stays bounded away from zero, say by $\lambda/2$, by making M smaller if necessary.

Hence we can talk about solutions u_0 and u_1 to the corresponding Dirichlet problem with the same boundary data g for \mathcal{L}_0 and \mathcal{L}_1 , respectively. Let $F = u_0 - u_1$. It follows that $\mathcal{L}_0 F = -\mathcal{L}_0 u_1$, so

$$F(x) = \int_D G_0(x, y) \mathcal{L}_0 u_1 dy.$$

Here G_0 is the Green's function of the operator \mathcal{L}_0 .

We will use the following two lemmas and defer their proofs until later. The following is analogous to Lemma 2.9 in [6].

Lemma 5.1. *There exists a constant $C = C(\lambda, n)$ such that under the hypotheses of Theorem 3.1,*

$$\tilde{N}F(Q) + \tilde{N}(\delta |\nabla F|)(Q) \leq C\varepsilon_0 M_{\omega_0}(A_{\tilde{\alpha}}u_1)(Q).$$

The next lemma is analogous to Lemma 2.16 in [6].

Lemma 5.2. *Let $\alpha > 0$. Then there exists $0 < \beta < \alpha$ depending only on the dimension, the number α and the Lipschitz constant of the domain D such that the following holds:*

Suppose that $S_\beta(F)(P) \leq \lambda$ for some P in a surface ball $\Delta = \Delta(P_0, r) \subset \partial D$. Then there exists $c > 0$, $\delta > 0$ depending only on the Lipschitz character of the domain D and the ellipticity constant of the operator \mathcal{L}_0 such that for any $\gamma > 0$

$$\sigma(\{Q \in \Delta \ ; \ S_\beta(F) > 2\lambda, \tilde{N}_\alpha(F) \leq \gamma\lambda, \tilde{N}_\alpha(\delta |\nabla F|) \leq \gamma\lambda, \tilde{N}_\alpha(F)A_\alpha(u_1) \leq (\gamma\lambda)^2\}) \leq c\gamma^\delta \sigma(\Delta). \quad (5.9)$$

Assuming Lemmas 5.1 and 5.2, we have

$$\begin{aligned} \int_{\partial B_1} \tilde{N}(F)^p d\sigma &\leq \int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma \\ &\leq C\varepsilon_0 \int_{\partial B_1} (M_{\omega_0}(A_{\tilde{\alpha}}u_1))^p d\sigma \\ &\leq C\varepsilon_0 \int_{\partial B_1} (M_{\omega_0}(A_{\tilde{\alpha}}u_1))^p k_0^{-1} d\omega_0 \\ &\leq C'\varepsilon_0 \int_{\partial B_1} A_{\tilde{\alpha}}(u_1)^p k_0^{-1} d\omega_0, \end{aligned}$$

where $k_0 = \frac{d\sigma}{d\omega_0}$. Recall that $k_0 \in RH_{p'}(d\sigma)$ (since we are assuming \mathcal{D}_p solvability for \mathcal{L}_0) is equivalent to $k_0^{-1} \in A_p(d\omega_0)$.

We then have

$$\begin{aligned} \int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma &\leq C\varepsilon_0 \int_{\partial B_1} A_{\tilde{\alpha}}(u_1)^p d\sigma \\ &\leq C'\varepsilon_0 \int_{\partial B_1} S_{c\tilde{\alpha}}(u_1)^p d\sigma \\ &\leq C''\varepsilon_0 \int_{\partial B_1} S_\beta(u_1)^p d\sigma \\ &\leq C'''\varepsilon_0 \int_{\partial B_1} (S_\beta(u_0)^p + S_\beta(F)^p) d\sigma, \end{aligned} \quad (5.10)$$

where we used Theorem 2.19 from [10] and also Theorem 2.17 of [10] for

$\|S_{c\tilde{\alpha}}(u_1)\|_{L^p} \approx \|S_\beta(u_1)\|_{L^p}$. It remains to deal with these terms.

First, we note that $\int_{\partial B_1} (S_{c\tilde{\alpha}}(u_0))^p d\sigma \lesssim \int_{\partial B_1} f^p d\sigma$, using Theorem 2.17 from [10] and our assumption that the Dirichlet problem is solvable for \mathcal{L}_0 .

For the second term, we will use the good-lambda inequality from Lemma 5.2, more specifically its Corollary 8.1. According to it, we have an estimate

$$\int_{\partial B_1} S_\beta(F)^p d\sigma \leq 2C \int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma + \int_{\partial B_1} S_\beta(u_0)^p d\sigma. \quad (5.11)$$

The term $\int_{\partial B_1} S_\beta(u_0)^p d\sigma$ is harmless and can be estimated by $\int_{\partial B_1} f^p d\sigma$, as above. Now we put (5.10) and (5.11) together to obtain

$$\int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma \leq C\varepsilon_0 \int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p + f^p \right) d\sigma.$$

Hence for ε_0 sufficiently small so that $C\varepsilon_0 \leq 1/2$, we have that

$$\int_{\partial B_1} \left(\tilde{N}(F)^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma \leq C \int_{\partial B_1} f^p d\sigma.$$

This is the estimate required for the solvability of the Dirichlet problem \mathcal{D}_p for \mathcal{L}_1 , as $u_1 = u_0 - F$, and for u_0 we have the needed estimates due to the L^p solvability for \mathcal{L}_0 . Since for $\alpha' < \alpha$ we have a pointwise estimate:

$$N_{\alpha'} u_1(Q) \lesssim \tilde{N}_\alpha u_1(Q) + \tilde{N}_\alpha(\delta |\nabla u_1|)(Q),$$

the theorem follows.

6. PROOF OF THEOREM 3.2

Now that we have established the perturbation theorem, we can easily dispense with the proof of Theorem 3.2. The first part of the proof, dealing with the smooth perturbation of A , is exactly the same as the smooth perturbation part of the proof of Corollary 2.3 in [4]. We repeat it here for convenience.

We prove this in the flat case; the general result will follow from a change of variables by Nečas and Stein (see, e.g., p.2 of [4] for details). The notation $\text{avg}(a)$ at a point (y, s) represents the average of a over the ball of radius $s/2$ centered at (y, s) (denoted $B_{s/2}(y, s)$). Given a matrix coefficient $a(x, t)$ in \mathbb{R}_+^n , set $\tilde{a}(x, t) = \int a(u, s) \phi_t(x - u, s - t) ds du$, where ϕ is a smooth bump function supported in the ball of radius $1/2$ and $\phi_t(y, s) = t^{-n} \phi(y/t, s/t)$.

We are assuming that

$$\left(\sup \left\{ |a(y, s) - \text{avg}(a(x, t))|^2 : (y, s) \in B_{t/2}(x, t) \right\} \right) \frac{dx dt}{t} \quad (6.12)$$

is a Carleson measure with small norm.

We aim to establish three facts:

$$t |\nabla \tilde{a}(x, t)|^2 dx dt \quad (6.13)$$

is a Carleson measure with small norm,

$$\left(\sup \left\{ |a(y, s) - \tilde{a}(y, s)|^2 : (y, s) \in B_{t/2}(x, t) \right\} \right) \frac{dx dt}{t} \quad (6.14)$$

satisfies the hypotheses of Theorem 3.1, and

$$a(x, t) \in \text{BMO}_\varrho \text{ for } \varrho = \varrho(M), \text{ with } \varrho \rightarrow 0 \text{ as } M \rightarrow 0. \quad (6.15)$$

Given the results in [4], the condition (6.13) implies that \mathcal{D}_p holds for the operator with coefficients \tilde{A} . Using (6.15), if M is chosen sufficiently small we will have $\varrho < \varrho_0$. Combining this with (6.14), as in the proof of Theorem 3.1 above, yields that \tilde{A} is in $\text{BMO}_{\tilde{\varrho}}$, for some $\tilde{\varrho} < \varrho_0$. Thus, the hypotheses for Theorem 3.1 are satisfied and, therefore, \mathcal{D}_p holds for the operator with coefficients A .

That (6.13) follows from the hypotheses of Theorem 3.2 is a straightforward calculation; apply the gradient to $\phi_t(y, s)$ and subtract a constant from the a_{ij} inside the integrand to see that

$$|\nabla \tilde{a}(x, t)| \leq Ct^{-1} \left(\sup \left\{ |a(y, s) - \text{avg}(a(x, t))| : (y, s) \in B_{t/2}(x, t) \right\} \right).$$

The proof of (6.14) is equally straightforward; add and subtract the constant $\text{avg}(a(x, t))$ inside the difference. For precise details see [4] and a similar calculation in [10].

It only remains to prove (6.15). Choose an arbitrary point (x, t) in our domain. We shall check that the function a is BMO near this point. Consider a ball B of radius $s > 0$ centered at the point (x, t) . There are three cases to consider:

- (i) a small ball, with $s < t/2$,
- (ii) a large ball, with $s \geq 2t$,
- (iii) an intermediate ball, with $t/2 \leq s < 2t$.

As we shall see, only the cases (i) and (ii) are fundamental. Case (iii) is merely a combination of the approaches taken in (i) and (ii).

In case (i), (6.12) trivially gives that

$$\sup_{(y, u) \in B_{t/8}(x, t)} |a(y, u) - \text{avg}(a(x, t))|^2 \lesssim M,$$

hence

$$\text{osc}_{B_{t/8}(x, t)} a = \max_{ij} \left| \sup_{(y, u) \in B_{t/8}(x, t)} a^{ij}(y, u) - \inf_{(y, u) \in B_{t/8}(x, t)} a^{ij}(y, u) \right| \lesssim M^{1/2},$$

From this

$$\text{osc}_{B_s(x, t)} a \leq \text{osc}_{B_{t/2}(x, t)} a \lesssim M^{1/2} \quad (6.16)$$

as the ball of radius $t/2$ can be covered by a fixed number (depending only on dimension) of balls of radius $t/8$. This immediately gives

$$|B_s(x, t)|^{-1} \int_{B_s(x, t)} |a^{ij}(y, u) - \text{avg}_{B_u}(a^{ij})| dy du \lesssim M^{1/2},$$

hence $\varrho \leq CM^{1/2}$.

If (ii) holds then $B_s(x, t)$ intersects the boundary $\{t = 0\}$ at a large set of area of order s^{n-1} . One might think of $D \cap B_s(x, t)$ as a subset of a larger Carleson box $T(\Delta)$, where Δ is a surface ball on the boundary $\{t = 0\}$ of radius comparable to s (a multiple of s where the constant depends on the dimension of our domain). Therefore, it will suffice to prove that

$$\int_{T(\Delta)} |a^{ij}(x, t) - \text{avg}_{T(\Delta)}(a^{ij})| dx dt \lesssim M^{1/2} s^n,$$

from which again $\varrho \leq CM^{1/2}$ on such balls.

In fact, the exact average that gets subtracted off in the BMO norm does not matter, so we might as well prove that

$$\int_{T(\Delta)} |a^{ij}(x, t) - \text{avg}_{T_0(\Delta)}(a^{ij})| dx dt \lesssim M^{1/2} s^n.$$

Here $T_k(\Delta)$, $k = 0, 1, 2, \dots$ is defined diadically by

$$T_k(\Delta) = \{(x, t) \in T(\Delta); t \in (2^{-k-1}s, 2^{-k}s]\}.$$

It follows that $T(\Delta)$ is a disjoint union of $T_k(\Delta)$, $k \geq 0$.

By (6.16) we immediately get that

$$|a^{ij}(x, t) - \text{avg}_{T_0(\Delta)}(a^{ij})| \lesssim M^{1/2}, \quad \text{for all } (x, t) \in T_0(\Delta).$$

Now consider $(x, t) \in T_1(\Delta)$. By using (6.16) twice we get that

$$\left| a^{ij}(x, t) - \text{avg}_{T_0(\Delta)}(a^{ij}) \right| \lesssim 2M^{1/2}, \quad \text{for all } (x, t) \in T_1(\Delta),$$

and inductively

$$\left| a^{ij}(x, t) - \text{avg}_{T_0(\Delta)}(a^{ij}) \right| \lesssim (k+1)M^{1/2}, \quad \text{for all } (x, t) \in T_k(\Delta).$$

Hence,

$$\begin{aligned} \int_{T(\Delta)} \left| a^{ij}(x, t) - \text{avg}_{T_0(\Delta)}(a^{ij}) \right| dx dt &= \\ \sum_{k=0}^{\infty} \int_{T_k(\Delta)} \left| a^{ij}(x, t) - \text{avg}_{T_0(\Delta)}(a^{ij}) \right| dx dt & \\ \lesssim \sum_{k=0}^{\infty} (k+1)M^{1/2}(2^{-k}s^n) &\approx M^{1/2}s^n, \end{aligned}$$

since $|T_k(\Delta)| \approx 2^{-k}s^n$.

Case (iii) is a combination of these two approaches where one considers the integrals on pieces $B_s(x, t) \cap \{(y, u); u \in (2^{-k-1}t, 2^{-k}t]\}$. We leave the details for the reader.

By combining (i), (ii) and (iii) we see that $a^{ij} \in \text{BMO}_\varrho$ for $\varrho \leq CM^{1/2}$, where M is the bound on the Carleson measure of (6.12) and $C > 0$ is a constant that depends on the dimension of our domain D .

7. PROOF OF LEMMA 5.1

Proof of Lemma 5.1. From [10], Lemma 3.2, we know that

$$\tilde{N}_\alpha F(Q) \lesssim \varepsilon_0 M_{\omega_0}(A_{\tilde{\alpha}} u_1)(Q) \quad (7.17)$$

for some $\tilde{\alpha}$ slightly larger than α . We will also show that

$$\begin{aligned} &\left(\tilde{N}_\alpha(\delta |\nabla F|) \right)^2(Q) \\ &\lesssim \tilde{N}_{\tilde{\alpha}}(F)(Q) \tilde{N}_{\tilde{\alpha}}(\delta |\nabla F|)(Q) + \left(\tilde{N}_{\tilde{\alpha}}(F) \right)^2(Q) + \varepsilon_0 \tilde{N}_{\tilde{\alpha}}(F)(Q) A_{\tilde{\alpha}}(u_1)(Q). \end{aligned} \quad (7.18)$$

Combining these two yields the lemma. Thus it remains to show (7.18).

To this end, we fix $Q \in \partial D$, $x \in \Gamma_\alpha(Q)$. Also, find the required value for r_0 in Lemma 2.14 of [10] (if necessary, making $r_0 < \frac{1}{4}$), and then choose $r^* \leq r_0/2$, where r^* is the truncation level of $\Gamma_\alpha(Q)$.

Under these assumptions, if we take $y \in B_0 = B(x, \frac{\delta(x)}{6})$ then $y \in \Gamma_{\tilde{\alpha}}(Q)$ for a slightly larger cone and also $y \in \partial B_r(Q)$ for $r \leq r_0$. Hence, Lemma 2.14 of [10] can be applied to all of the points in our integral. Lemma 2.14 in [10] provides the estimate:

$$\frac{G_0(0, y)}{\omega_0(\Delta_r(Q))} \simeq \frac{\delta^2(y) \wp(y)}{\wp(B(y))},$$

where $r = |y - Q|$. For $y \in B_0$, we have that $\frac{5}{6}\delta(x) \leq r \leq \frac{7}{6}\delta(x)$. We observe that for r in this range, all values of $\omega_0(\Delta_r(Q))$ are comparable to the value of $\omega_0(\Delta_{\delta(x)}(Q))$, as the measure is doubling. Also, let $\delta := \delta(x)$.

Following [6], we start with

$$\int_{B_0} \delta^2(y) |\nabla F(y)|^2 \frac{\wp(y)}{\wp(B(y))} dy \lesssim \frac{1}{\delta} \int_{\delta/6}^{\delta/5} \int_{B_r} \delta^2(y) |\nabla F(y)|^2 \frac{\wp(y)}{\wp(B(y))} dy dr,$$

where $B_r = B(x, r)$. However, we need to average twice. Thus, we estimate

$$\int_{B_0} |\nabla F(y)|^2 \frac{\delta^2(y) \wp(y)}{\wp(B(y))} dy \lesssim \frac{1}{\delta^2} \int_0^\delta \int_{\varrho_1(s)}^{\varrho_2(s)} \int_{B_r} |\nabla F(y)|^2 \frac{\delta^2(y) \wp(y)}{\wp(B(y))} dy dr ds, \quad (7.19)$$

with $\varrho_1(s) = (\beta_1 - \frac{1}{6})s + \frac{\delta}{6}$, $\varrho_2(s) = (\beta_2 - \frac{1}{5})s + \frac{\delta}{5}$, with $\beta_1 < \frac{1}{6} < \frac{1}{5} < \beta_2$. The β_i 's are yet to be determined.

Then,

$$\begin{aligned} \int_{B_r} \delta^2(y) |\nabla F(y)|^2 \frac{\wp(y)}{\wp(B(y))} dy &\simeq \int_{B_r} |\nabla F(y)|^2 \frac{G_0(0, y)}{\omega_0(\Delta_r(Q))} dy \\ &\lesssim \frac{1}{\omega_0(\Delta_\delta(Q))} \int_{B_r} A_0 \nabla F \cdot \nabla F G_0(0, y) dy \\ &\simeq \frac{1}{\omega_0(\Delta_\delta(Q))} \int_{B_r} (\mathcal{L}_0(F^2) - 2F \mathcal{L}_0 F) G_0(0, y) dy \\ &:= I_1 + I_2. \end{aligned}$$

Here A_0 is the matrix of coefficients (a_0^{ij}) . We first estimate the contribution to (7.19) by I_1 . Integration by parts twice yields:

$$\begin{aligned} I_1 \omega_0(\Delta_\delta(Q)) &= \int_{B_r} \mathcal{L}_0(F^2) G_0(0, y) dy \\ &= \sum \int_{\partial B_r} G_0 a_0^{ij} \partial_j(F^2) \nu_i d\sigma - \sum \int_{B_r} \partial_i(G_0 a_0^{ij}) \partial_j(F^2) dy \\ &= \sum \int_{\partial B_r} \left(G_0 a_0^{ij} \partial_j(F^2) \nu_i - \partial_i(a_0^{ij} G_0) F^2 \nu_j \right) d\sigma + \int_{B_r} \mathcal{L}_0^*(G_0) F^2 dy. \end{aligned}$$

However, $\mathcal{L}_0^*(G_0) = 0$, so we are only left with the two boundary terms. Hence,

$$\begin{aligned} &\frac{1}{\delta} \int_{\varrho_1(s)}^{\varrho_2(s)} I_1 dr \\ &= \frac{1}{\delta \omega_0(\Delta_\delta(Q))} \sum \left[\int_{B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}} \left(G_0 a_0^{ij} \partial_j(F^2) \nu_i - \partial_i(a_0^{ij} G_0) F^2 \nu_j \right) dy \right] \\ &= \frac{1}{\delta \omega_0(\Delta_\delta(Q))} \sum \left[\int_{B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}} G_0 a_0^{ij} (\partial_j(F^2) \nu_i + \partial_i(F^2) \nu_j + F^2 \partial_i \nu_j) dy \right. \\ &\quad \left. - \int_{\partial(B_{\varrho_2(s)} \setminus B_{\varrho_1(s)})} G_0 a_0^{ij} F^2 \nu_i \nu_j d\sigma \right] \end{aligned}$$

We estimate this term by term, starting with

$$\begin{aligned} & \frac{1}{\delta \omega_0(\Delta_\delta(Q))} \sum \int_{B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}} \left| G_0 a_0^{ij} (\partial_j(F^2) \nu_i + \partial_i(F^2) \nu_j) \right| dy \\ & \lesssim \frac{1}{\delta} \int_{B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}} |F| |\nabla F| \frac{\delta^2 \wp(y)}{\wp(B(y))} dy := I_1^a \end{aligned}$$

And

$$\begin{aligned} \frac{1}{\delta} \int_0^\delta I_1^a ds & \lesssim \int_{B_{\beta_2 \delta} \setminus B_{\beta_1 \delta}} |F| |\nabla F| \frac{\delta \wp(y)}{\wp(B(y))} dy \\ & \lesssim \tilde{N}_{\tilde{\alpha}}(F) \tilde{N}_{\tilde{\alpha}}(\delta |\nabla F|). \end{aligned} \quad (7.20)$$

Recall that $\tilde{\alpha}$ must be chosen a little larger than α . The parameters β_i determine the size of $\tilde{\alpha}$, as we want all points in $B_{\beta_2 \delta} \subset \Gamma_{\tilde{\alpha}}(Q)$. The this choice is irrelevant as long as $\Gamma_{\tilde{\alpha}}(Q)$ is still a nontagential cone.

Next, we look at ν , the outward unit normal for B_r arising from our first integration by parts. We know $\nu_j = \frac{x_j}{|x|}$ when $|x| = r$, and $\partial_i(\nu_j) = \frac{x_i x_j}{|x|^3}$. Thus, for $x \in B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}$, $\beta_1 \delta \leq r = |x| \leq \beta_2 \delta$ whence $|\partial_i \nu_j| \lesssim \frac{1}{\delta}$. This leads to

$$\begin{aligned} & \frac{1}{\delta \omega_0(\Delta_\delta(Q))} \sum \int_{B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}} \left| G_0 a_0^{ij} F^2 \partial_i(\nu_j) \right| dy \\ & \lesssim \frac{1}{\delta^2} \int_{B_{\varrho_2(s)} \setminus B_{\varrho_1(s)}} F^2 \frac{\delta^2 \wp(y)}{\wp(B(y))} dy := I_1^b, \end{aligned}$$

and

$$\frac{1}{\delta} \int_0^\delta I_1^b ds \lesssim \int_{B_{\beta_2 \delta} \setminus B_{\beta_1 \delta}} F^2 \frac{\wp(y)}{\wp(B(y))} dy \lesssim \left(\tilde{N}_{\tilde{\alpha}}(F) \right)^2. \quad (7.21)$$

For the last term, we see

$$\begin{aligned} & \frac{1}{\delta^2 \omega_0(\Delta_\delta(Q))} \sum \int_0^\delta \int_{\partial(B_{\varrho_2(s)} \setminus B_{\varrho_1(s)})} \left| G_0 a_0^{ij} F^2 \nu_i \nu_j \right| d\sigma ds \\ & \lesssim \frac{1}{\delta^2 \omega_0(\Delta_\delta(Q))} \int_{(B_{\beta_2 \delta} \setminus B_{\delta/5}) \cup (B_{\delta/6} \setminus B_{\beta_1 \delta})} G_0 F^2 dy \\ & \lesssim \int_{B_{\beta_2 \delta}} F^2 \frac{\wp(y)}{\wp(B(y))} dy \lesssim \left(\tilde{N}_{\tilde{\alpha}}(F) \right)^2. \end{aligned} \quad (7.22)$$

We now turn to estimating I_2 ; by the fact that ω is doubling,

$$|I_2| \lesssim \int_{B_r} |F| |\mathcal{L}_0 F| \frac{G_0(0, y)}{\omega_0(\Delta_\delta(Q))} dy \lesssim \varepsilon_0 \int_{B_r} |F| |\nabla^2 u_1| \frac{\delta^2(y) \wp(y)}{\wp(B(y))} dy. \quad (7.23)$$

Here we are using the fact that $\mathcal{L}_0 F = -\mathcal{L}_0 u_1 = -(\mathcal{L}_1 + \varepsilon^{ij} \partial_{ij}) u_1 = -\varepsilon^{ij} \partial_{ij} u_1$ and $\sup_{z \in B_r} |\varepsilon^{ij}(z)| \leq \sup_{z \in B_{(x, \delta/2)}} |\varepsilon(z)| = \mathbf{a}(x)$. Using the condition from Theorem 3.1, we get that $\mathbf{a}(x) \lesssim \varepsilon_0$.

Thus,

$$\begin{aligned} \frac{1}{\delta^2} \int_0^\delta \int_{\varrho_1(s)}^{\varrho_2(s)} |I_2| \, dr \, ds &\lesssim \frac{1}{\delta} \int_{\beta_1 \delta}^{\beta_2 \delta} |I_2| \, dr \\ &\lesssim \varepsilon_0 \int_{B_{\beta_2 \delta}} |F| |\nabla^2 u_1| \frac{\delta^2(y) \wp(y)}{\wp(B(y))} \, dy \lesssim \varepsilon_0 \tilde{N}_{\tilde{\alpha}}(F) A_{\tilde{\alpha}}(u_1) \end{aligned} \quad (7.24)$$

By combining (7.19), (7.20), (7.21), (7.22) and (7.24), the lemma is proven. \square

8. PROOF OF LEMMA 5.2

We now prove Lemma 5.2, the good-lambda inequality which is crucial for estimating $S(F)$.

Proof of Lemma 5.2. Let us call E the set

$$\begin{aligned} E = \left\{ Q \in \Delta; S_\beta(F) > 2\lambda, \tilde{N}_\alpha(F) \leq \gamma\lambda, \right. \\ \left. \tilde{N}_\alpha(\delta |\nabla F|) \leq \gamma\lambda, \tilde{N}_\alpha(F) A_\alpha(u_1) \leq (\gamma\lambda)^2 \right\}. \end{aligned}$$

It is sufficient to prove that $\omega(E) \leq C\gamma^2\omega(\Delta)$, since we already know that $\omega \in A_\infty(d\sigma)$. Standard arguments (see [3] or [6]) show that since $S_\beta(F) > 2\lambda$ on E , we can choose $\gamma > 0$ sufficiently small so that $S_{\beta, \tau r}(F) > \lambda/2$, where each cone is truncated at height τr for some fixed $0 < \tau < 1$. It follows that

$$\omega(E) \leq \frac{4}{\lambda^2} \int_E S_{\beta, \tau r}^2(F)(Q) d\omega(Q). \quad (8.25)$$

We will introduce the following notation. Let

$$\mathcal{D} := \bigcup_{Q \in E} \Gamma_{\beta, \tau r}(Q). \quad (8.26)$$

For $\alpha' \in (\beta, \alpha)$, let us consider a smoothed-out version of the set $\bigcup_{Q \in E} \Gamma_{\alpha', \tau r}(Q)$. We denote by $\mathcal{D}_{\alpha'}$ the set with the properties:

- (i) $\mathcal{D} \subset \bigcup_{Q \in E} \Gamma_{\alpha', \tau r}(Q) \subset \mathcal{D}_{\alpha'} \subset \bigcup_{Q \in E} \Gamma_{\alpha, 3\tau r}(Q)$,
- (ii) $\partial \mathcal{D}_{\alpha'}$ is smooth except at E and $|\nabla \nu(Q)| \leq C/\delta_D(Q)$ for $Q \in \partial \mathcal{D}_{\alpha'}$,
- (iii) $\mathcal{D}_{\alpha'} \subset \mathcal{D}_{\alpha''}$ if $\alpha' < \alpha''$.

Here ν is the outer normal at the boundary and $\delta = \delta_D$ denotes the distance to the boundary of the original domain D . We now work with (8.25).

$$\begin{aligned} \omega(E) &\leq \frac{4}{\lambda^2} \int_E \left(\int_{\Gamma_{\beta, \tau r}(Q)} |\nabla F|^2 \frac{\delta^2 \wp(x)}{\wp(B(x))} dx \right) d\omega(Q) \\ &\leq \frac{C}{\lambda^2} \int_E \left(\int_{\Gamma_{\beta, \tau r}(Q)} |\nabla F|^2 G_0 \frac{dx}{\omega(\Delta_x)} \right) d\omega(Q) \\ &\leq \frac{C}{\lambda^2} \int_{\mathcal{D}_{\alpha'}} |\nabla F|^2 G_0 \, dx \\ &\leq \frac{C}{\lambda^2} \int_{\mathcal{D}_{\alpha'}} (A_0 \nabla F \cdot \nabla F) G_0 \, dx. \end{aligned} \quad (8.27)$$

Here $\Delta_x = \{Q \in \partial D; x \in \Gamma_\beta(Q)\}$ and $\alpha' \in (\beta, \alpha)$. Now,

$$A_0 \nabla F \cdot \nabla F = \mathcal{L}_0(F^2) - 2F \mathcal{L}_0 F,$$

so there are two terms to estimate

$$\begin{aligned} & \int_{\mathcal{D}_{\alpha'}} (A_0 \nabla F \cdot \nabla F) G_0 dx \\ &= \int_{\mathcal{D}_{\alpha'}} \mathcal{L}_0(F^2) G_0 dx - \int_{\mathcal{D}_{\alpha'}} 2F \mathcal{L}_0 F G_0 dx. \end{aligned} \quad (8.28)$$

Let us denote these two terms by I_1 and I_2 . We first deal with I_1 . Recall that $\mathcal{L}_0 G_0 = -\delta(0)$, hence integration by parts gives us only two boundary terms

$$I_1 \leq \int_{\partial \mathcal{D}_{\alpha'}} a_0^{ij} \partial_i (F^2) G_0 \nu_j d\sigma - \int_{\partial \mathcal{D}_{\alpha'}} \partial_j (a_0^{ij} G_0) F^2 \nu_i d\sigma. \quad (8.29)$$

Note that, strictly speaking, these two boundary terms are not well-defined. To fix this, we again use the averaging technique introduced before. We integrate over the interval $[\alpha', \alpha''] \subset (\beta, \alpha)$ and get instead solid integrals

$$\begin{aligned} I_1 &\leq c \left| \int_{\mathcal{D}_{\alpha''} \setminus \mathcal{D}_{\alpha'}} a_0^{ij} \partial_i (F^2) G_0 \delta^{-1} \nu_j dx \right| \\ &\quad + c \left| \int_{\mathcal{D}_{\alpha''} \setminus \mathcal{D}_{\alpha'}} \partial_j (a_0^{ij} G_0) F^2 \delta^{-1} \nu_i dx \right| \\ &= I_1^a + I_1^b. \end{aligned} \quad (8.30)$$

Now, for simplicity let $\tilde{\mathcal{D}} = \mathcal{D}_{\alpha''} \setminus \mathcal{D}_{\alpha'}$. We see that

$$\begin{aligned} I_1^a &\leq C \int_{\tilde{\mathcal{D}}} |F| |\nabla F| G_0 \delta^{-1} dx \\ &\approx \int_{Q \in 2\Delta} \left(\int_{\Gamma_\beta(Q) \cap \tilde{\mathcal{D}}} |F| |\nabla F| G_0 \frac{dx}{\delta \omega(\Delta_x)} \right) d\omega(Q) \\ &\approx \int_{Q \in 2\Delta} \left(\int_{\Gamma_\beta(Q) \cap \tilde{\mathcal{D}}} |F| |\nabla F| \delta \frac{\wp(x)}{\wp(B(x))} dx \right) d\omega(Q). \\ &\leq \int_{Q \in 2\Delta} \left(\int_{\Gamma_\beta(Q) \cap \tilde{\mathcal{D}}} |F| \frac{\wp(x)}{\wp(B(x))} dx \right)^{1/2} \times \\ &\quad \left(\int_{\Gamma_\beta(Q) \cap \tilde{\mathcal{D}}} |\nabla F|^2 \delta^2 \frac{\wp(x)}{\wp(B(x))} dx \right)^{1/2} d\omega(Q). \end{aligned} \quad (8.31)$$

The key is that if $x \in \Gamma_\beta(Q) \cap \tilde{\mathcal{D}}$ then $x \in \Gamma_\alpha(Q')$ for some $Q' \in E$ and the set $\Gamma_\beta(Q) \cap \tilde{\mathcal{D}}$ is of diameter proportional to $\delta(x)$ and its distance to ∂D is also of $\delta(x)$ size. This implies we can control the two solid integrals the the last line by $\tilde{N}_\alpha(F)(Q') \tilde{N}_\alpha(\delta |\nabla F|)(Q')$. This gives

$$I_1^a \leq C \int_{Q \in 2\Delta} (\gamma \lambda)^2 d\omega(Q) = C \gamma^2 \lambda^2 \omega(\Delta),$$

since the measure ω is doubling. To estimate I_1^b we integrate by parts one more time. We get

$$I_1^b \leq c \left| \int_{\tilde{\mathcal{D}}} a_0^{ij} G_0 \partial_j \left(\frac{F^2 \nu_i}{\delta} \right) dx \right| + c \left| \int_{\partial \tilde{\mathcal{D}}} a_0^{ij} G_0 F^2 \nu_i \nu_j \delta^{-1} d\sigma \right|. \quad (8.32)$$

The first term of (8.32) will give us two additional terms, depending on where the derivative ∂_j falls. By the chain rule,

$$\left| \partial_i \left(\frac{F^2 \nu_i}{\delta} \right) \right| \leq \frac{C |F| |\nabla F|}{\delta} + \frac{C F^2}{\delta^2}.$$

Here we use the fact that the real distance function δ can be replaced by a smooth distance function so that $|\nabla \delta^{-1}| \approx \delta^{-2}$ and also $|\nabla \nu_i| \leq C \delta^{-1}$. Hence, the first term is of the same type as I_1^a , and the second one can be bounded by

$$c \int_{\tilde{\mathcal{D}}} F^2 G_0 \delta^{-2} dx \approx \int_{Q \in 2\Delta} \left(\iint_{\Gamma_\beta(Q) \cap \tilde{\mathcal{D}}} |F|^2 \delta \frac{\wp(x)}{\wp(B(x))} dx \right) d\omega(Q). \quad (8.33)$$

Thus, this term can be dominated by $C \int_{Q \in 2\Delta} \left(\tilde{N}_\alpha(F)(Q') \right)^2 d\sigma \leq C \gamma^2 \lambda^2 \omega(\Delta)$. Finally, (8.32) has one additional boundary term, which again has to be averaged out. So we need to use the wiggling technique one more time. Without going into too much detail, this will again turn the surface integral into a solid integral over a set we call $\tilde{\mathcal{D}}'$ (essentially of the same type as $\tilde{\mathcal{D}}$):

$$\left| \int_{\tilde{\mathcal{D}}'} a_0^{ij} G_0 \frac{F^2 \nu_i \nu_j}{\delta^2} dx \right| \lesssim \int_{\tilde{\mathcal{D}}'} F^2 G_0 \delta^{-2} dx.$$

Notice that this term is similar to (8.33), so the same estimates can be applied. This establishes

$$|I_1^a| \leq C \gamma^2 \lambda^2 \omega(\Delta).$$

Now we deal with I_2 . As before, we use $\mathcal{L}_0 F = -\varepsilon^{ij} \partial_{ij} u_1$, where $\varepsilon^{ij} = a_0^{ij} - a_1^{ij}$. This gives

$$I_2 \leq C \int_{\mathcal{D}_{\alpha'}} \varepsilon(x) |F| |\nabla^2 u_1| G_0 dx, \quad (8.34)$$

where $\varepsilon(x) = \max |\varepsilon_{ij}(x)|$. We turn this back (by Fubini) to into two integrals

$$\begin{aligned} I_2 &\lesssim \int_{Q \in 2\Delta} \left(\int_{\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}} \varepsilon(x) |F| |\nabla^2 u_1| \frac{G_0}{\omega(\Delta_x)} dx \right) d\sigma(Q) \\ &\approx \int_{Q \in 2\Delta} \left(\int_{\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}} \varepsilon(x) |F| |\nabla^2 u_1| \delta^2 \frac{\wp(x)}{\wp(B(x))} dx \right) d\sigma(Q). \end{aligned} \quad (8.35)$$

By Hölder:

$$\begin{aligned} I_2 &\lesssim \int_{Q \in 2\Delta} \left(\int_{\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}} \varepsilon(x)^2 |F|^2 \frac{\wp(x)}{\wp(B(x))} dx \right)^{1/2} \times \\ &\quad \left(\int_{\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}} |\nabla^2 u_1|^2 \delta^4 \frac{\wp(x)}{\wp(B(x))} dx \right)^{1/2} d\sigma(Q). \end{aligned} \quad (8.36)$$

As $\alpha' < \alpha$, it can be arranged that either $\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'} \subset \Gamma_\alpha(Q')$ for some $Q' \in E$ or $\tilde{N}_\alpha(F)(Q) A_\alpha(u_1)(Q) \leq (\gamma\lambda)^2$.

Indeed, if $Q \in \bar{E}$, then the fact that $\tilde{N}_\alpha(F)(Q_n) A_\alpha(u_1)(Q_n) \leq (\gamma\lambda)^2$ for a sequence of $Q_n \in E$ converging to Q implies the same for Q . In this case we just take $Q' = Q$.

Otherwise $d = \text{dist}(Q, E) > 0$, and this gives that $\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}$ only contains points of distance $\delta \gtrsim d$. Hence by making α sufficiently large we will have $\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'} \subset \Gamma_{\alpha}(Q')$ for all points $Q' \in E$ such that $\text{dist}(Q, Q') \approx d$. It follows that

$$\left(\int_{\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}} |\nabla^2 u_1|^2 \delta^4 \frac{\wp(x)}{\wp(B(x))} dx \right)^{1/2} \leq A_{\alpha}(u_1)(Q').$$

On the other hand,

$$\left(\iint_{\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}} \varepsilon(x)^2 |F|^2 \frac{\wp(x)}{\wp(B(x))} dx \right)^{1/2} \leq E_{3\tau r}(Q) \tilde{N}_{\alpha}(F)(Q'), \quad (8.37)$$

where

$$E_{3\tau r}(Q) = \left(\int_{\Gamma_{\alpha, 3\tau r}(Q)} \frac{\left(\sup_{B(x, \delta(x)/6)} \varepsilon(x) \right)^2}{\delta^n} dx \right)^{1/2}.$$

To see (8.37) we cover the set $\Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}$ by a union of balls of diadic diameters $2^k r$, $k \in \mathbb{Z}$, with each such ball of approximate distance $2^k r$ to the boundary such that each point $x \in \Gamma_{\alpha'}(Q) \cap \mathcal{D}_{\alpha'}$ belongs to at most K balls. (Simple geometric considerations imply that K will only depend on the dimension, the number α' and the Lipschitz constant of D). On each such ball, the square of the solid integral on the left-hand side of (8.37) can be estimated by $C \left(\sup_{x \in B^i} \varepsilon(x)^2 \right) \tilde{N}_{\alpha}(F)^2(Q')$. After we sum over all the balls we get the expression $CK E_{3\tau r}^2(Q) \tilde{N}_{\alpha}(F)^2(Q')$. It follows that

$$\begin{aligned} I_2 &\leq C \int_{Q \in 2\Delta} A_{\alpha}(u_1)(Q') \tilde{N}_{\alpha}(F)(Q') E_{3\tau r}(Q) d\omega(Q) \\ &\leq C \gamma^2 \lambda^2 \int_{Q \in 2\Delta} E_{3\tau r}(Q) d\omega(Q) \\ &\leq C \gamma^2 \lambda^2 \omega(2\Delta)^{1/2} \left(\int_{Q \in 2\Delta} E_{3\tau r}^2(Q) d\omega(Q) \right)^{1/2} \\ &\leq C \gamma^2 \lambda^2 \omega(2\Delta) \leq C' \gamma^2 \lambda^2 \omega(\Delta). \end{aligned} \quad (8.38)$$

since

$$\int_{Q \in 2\Delta} E_{3\tau r}^2(Q) d\omega(Q) \leq C \omega(2\Delta)$$

by Rios's work (see p. 683 of [10]). Note that this is the only place we are using (3.4), and we do not use the fact that it is small, only that it is finite. This establishes the good-lambda lemma. \square

Corollary 8.1. *Lemma 5.2 implies that for any $1 < p < \infty$:*

$$\int_{\partial D} S(F)^p d\sigma \leq C(q) \int_{\partial D} \left((\tilde{N}(F))^p + \tilde{N}(\delta |\nabla F|)^p \right) d\sigma + \int_{\partial D} S(u_0)^p d\sigma, \quad (8.39)$$

where the square function S is defined over cones of smaller aperture than the modified nontangential maximal function \tilde{N} .

Proof. Indeed, the Whitney decomposition and Lemma 5.2 gives us

$$\int_{\partial D} S_\beta(F)^p d\sigma \leq C \left[\int_{\partial D} \left(\tilde{N}_\alpha(F)^p + \tilde{N}_\alpha(\delta |\nabla F|)^p \right) d\sigma + \int_{\partial D} (A_\alpha(u_1) \tilde{N}_\alpha(F))^{p/2} d\sigma \right]$$

for some $\beta < \alpha$. This implies that for any $\varepsilon > 0$,

$$\int_{\partial D} S_\beta(F)^p d\sigma \leq C(\varepsilon) \int_{\partial D} \left(\tilde{N}_\alpha(F)^p + \tilde{N}_\alpha(\delta |\nabla F|)^p \right) d\sigma + \varepsilon \int_{\partial D} A_\alpha(u_1)^p d\sigma.$$

By Theorem 2.19 of [10], since u_1 is a solution to $\mathcal{L}_1 u_1 = 0$ we have pointwise estimate $A_\alpha(u_1) \leq C S_{c\alpha}(u_1)$ for some $c > 1$ depending only on the dimension n . Also by [5] (see also Theorem 2.17 of [10]) for solutions we have $\|S_{c\alpha}(u_1)\|_{L^p} \leq C \|S_\beta(u_1)\|_{L^p}$ with C only depending on the ellipticity constant, the numbers $c\alpha$ and β and the dimension.

This gives

$$\int_{\partial D} S_\beta(F)^p d\sigma \leq C(\varepsilon) \int_{\partial D} \left(\tilde{N}_\alpha(F)^p + \tilde{N}_\alpha(\delta |\nabla F|)^p \right) d\sigma + C_1 \varepsilon \int_{\partial D} S_\beta(u_1)^p d\sigma.$$

We can write $S_\beta(u_1)^p \leq C_2(S_\beta(u_0)^p + S_\beta(F)^p)$. Choose ε so that $C_1 C_2 \varepsilon < 1/2$ (this allows the term $C_1 C_2 \varepsilon S_\beta(F)^p$ to be incorporated into the right-hand side). It follows that

$$\int_{\partial D} S_\beta(F)^p d\sigma \leq 2C(\varepsilon) \int_{\partial D} \left(\tilde{N}_\alpha(F)^p + \tilde{N}_\alpha(\delta |\nabla F|)^p \right) d\sigma + \int_{\partial D} S_\beta(u_0)^p d\sigma.$$

□

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